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Discrete representations of the n-dimensional wave equation

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Abstract. A system of first-order difference equations on a rectangular n-dimensional lattice is presented, which reduces to the wave equation in the continuum limit. These equations allow solutions of the discrete wave equation to be expressed as summations of paths simpler than those obtained through standard path integral formalism, which in turn allows wave solutions to be simulated by the same Monte Carlo and other mehtods used to model diffusion phenomena.

1. Introduction

The general solution of the one-dimensional wave equation may be written as a sum of two functions, or travelling waves. The equations of motion of these travelling waves, first-order in time and space, are simple translations in opposite directions with a constant speed.

The second section of the paper extends the benefits of such an approach to higher dimensions so that, in general, the n-dimensional wave equation is likewise recast as a sum of $2n$ functions, first order in time, each of which implies a motion along the positive or negative directions of the n axes of the space.

The third section shows that this travelling wave decomposition lends itself to a statistical implementation. Recall that the kernel of the diffusion equation is expressible as the continuum limit of distributions of random walks on a lattice, and that the time evolution of the diffusion equation may be computed by ensembles of particles executing such random walks. By assigning these particles a (discrete) phase factor, it is possible to give the wave equation a similar implementation.

The third section shows how similar lattice methods may be used to model the n-dimensional Klein-Gordon equation.

2. The discrete wave equation

The discrete analogue of the wave equation is defined on an orthonormal, $(n+1)$-dimensional spacetime lattice of points whose spacing is unity. The second-order partial derivatives of time and space appearing in the d'Alembertian are replaced by their usual finite-difference analogues

$$\frac{\partial^2}{\partial x_i^2}\psi(\ldots, x_i, \ldots) \equiv \psi(\ldots, x_i+1, \ldots) - 2\psi(\ldots, x_i, \ldots) + \psi(\ldots, x_i-1, \ldots) \tag{1}$$

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where

$$x_i = x_1, x_2, \ldots, x_n, t.$$

Thus, the wave equation retains its usual form:

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi = \left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\ldots+\frac{\partial^2}{\partial x_n^2}\right)\psi. \tag{2}$$

Setting the constant c, which will be called the speed of light, to $\sqrt{1/n}$ confers several useful properties on the equations; among them, a conserved momentum and energy which in the continuum limit converge to their standard forms [1]—this is done throughout. In particular, because of the cancellation of all terms in the above equation corresponding to the central term of the right-hand side of (1), the spacetime is decoupled into two distinct lattices, even and odd (figure 1), such that every spacetime point (x, t) in the even (odd) sublattice has the property that

$$t+\sum_{i=1}^{n} x_i$$

is even (odd). Without loss of generality, it will be assumed that the solutions considered here are non-zero on only one of these lattices.

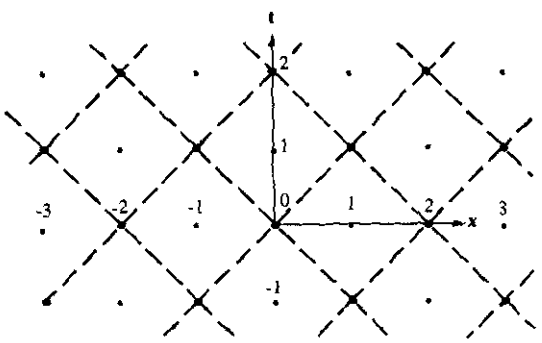

Figure 1. Two-dimensional discrete spacetime is decoupled into two distinct lattices, for the value of the speed of light chosen here. The diagonal arcs shown connect the nearest-neighbouring points of one such lattice. Higher dimensional spacetimes are likewise decoupled into two lattices.

It can be seen from the two preceding equations that solutions of the discrete wave equations form a vector space whose dimensionality is equal to twice the number of points in the lattice (i.e. the sublattice). An element of this space of solutions is completely determined by arbitrarily assigning values on all points of the lattice at two successive times.

Any solution of the n-dimensional discrete wave equation defined on an n-cubical lattice of N^n points may also be given in terms of harmonic solutions via Fourier analysis, e.g.

$$\psi(x, t) = \prod_{j=1}^{n}\sum_{\kappa_j=0}^{N-1} \mathcal{A}(\kappa)\exp\left(\frac{2\pi i}{N}(\kappa\cdot x+\omega t)\right) \tag{3}$$

where the $\mathscr{A}(\boldsymbol{\kappa})$ depend on the initial conditions. The frequency ω, whose functional dependence on the components of $\boldsymbol{\kappa}$ has been suppressed for clarity, is given by the dispersion relation

$$\sin^2\frac{\omega}{2}=\frac{1}{n}\left(\sin^2\frac{\kappa_1}{2}+\ldots+\sin^2\frac{\kappa_n}{2}\right). \tag{4}$$

This passes in the continuum limit to the familiar

$$\omega^2=\frac{1}{n}(k_1^2+\ldots+k_n^2) \tag{5}$$

where the k_m are the respective conjugate variables of the Fourier integral appearing in the continuum generalization of (3). Note that if $c^2=1/n$ as implied above, the numerical integration of (2) is stable. If one were to make the *a priori* simpler choice of setting $c^2=1$, or indeed to any value greater than $\sqrt{1/n}$, then for all except the one-dimensional case there would be spatial frequencies for which the modulus of the right-hand side of (4) would be greater than one. The time frequency ω would then have to contain an imaginary component in order for the equality to hold, leading in general to an exponential growth of the solutions. As it is, the solutions retain the characteristic undamped periodicity of their continuum analogues. Moreover, their deviation from the continuum case stays bounded. Of course, only solutions whose associated wavelengths are much larger than the lattice spacing can be well approximated by such discrete analogues, though in principle the correspondence may be made as close as desired.

2.1. The one-dimensional case

2.1.1. Travelling waves and arcs. As is well known, the general solution of the one-dimensional discrete wave equation may be written as the sum of two travelling waves,

$$\psi(x,t)=f_+(u)+f_-(v) \tag{6}$$

where these two travelling waves are, respectively, functions of the single variables $u=x-t$ and $v=x+t$, so that they translate unchanged in opposite directions as time passes.

Indeed, this propagation from one spacetime point to its neighbours suggests that these travelling waves are functions more naturally defined on the *arcs*, so to speak, connecting neighbouring spacetime points (figure 1), although the utility of such a view becomes fully manifest only in higher dimensions. Let $f_+^{\text{in}}(x,t)$ and $f_-^{\text{in}}(x,t)$ denote functions, which will be referred to as *flows*, defined on the arcs connecting spacetime points of the form $(x\mp1,\ t-1)$. Similarly, at the risk of redundancy, let $f_\pm^{\text{out}}(x,t)$ denote functions defined on arcs connecting spacetime points of the form $(x\pm1,\ t+1)$. Obviously,

$$f_\pm^{\text{in}}(x,t)\equiv f_\pm^{\text{out}}(x\mp1,\ t-1). \tag{7}$$

The decomposition of (6) then implies

$$f_\pm^{\text{in}}(x,t)=f_\pm^{\text{out}}(x,t). \tag{8}$$

Any solution of (2) may then be expressed in terms of functions defined on the arcs.

Like the set of point solutions, the set of arc solutions also forms a vector space, and the travelling wave decompsoition implies that there exists a homomorphism from

the arc solutions to the point solutions (figure 2). Moreover, this homomorphism is preserved over time, under the respective time evolution of each system. By the implicit use of this homomorphism, (6) may be restated as

$$\psi(x,t)=f_+^{\text{in}}(x,t)+f_-^{\text{in}}(x,t)=f_+^{\text{out}}(x,t)+f_-^{\text{out}}(x,t). \tag{9}$$

The homomorphism is obviously not one-to-one, since by everywhere setting

$$f_+^{\text{out}}(x,t)=-f_-^{\text{out}}(x,t)=\alpha$$

where α is an arbitrary constant, one obtains an infinity of arc solutions that under the homomorphism are mapped onto the trivial (i.e. everywhere vanishing) point solution.

By a proper choice of the basis functions, the expression of a given solution in terms of the flows can (in any number of dimensions) be made unique, so that the relation between the point solutions and arc solutions becomes an isomorphism.

2.1.2. Completeness. In considering the notion of completeness, for any number of dimensions, the space in question is assumed for convenience to be a torus of length $2N$ in all spatial dimensions, where N is, furthermore, an even number. Moreover, let all solutions have the property that

$$\psi(\ldots,x_i,\ldots)=-\psi(\ldots,x_i+N,\ldots) \tag{10}$$

where the '+' sign in the expression x_i+N implies addition modulo N. Therefore, a solution is completely determined by its boundary conditions on the subspace of points for which the x_i range from 0 to $N-1$. By making N as large as necessary, the desired generality is retained. (Choosing N to be even makes the initialization described below identical, up to a change of sign, for each subspace. Imposing the parity condition (10) excludes the unphysical or uninteresting solutions containing zero-frequency modes or terms linear in the time or space variables.)

Returning again to one-dimensional systems, let $h_\pm(x-x_0)$ stand for the travelling wave solutions that at $t=t_0$ are zero everywhere except on the arcs going out from the point x_0 to the respective point $x_0\pm1$, where they are 1. Such solutions, and their n-dimensional generalizations, will be called *hodotic* solutions (from the Greek word for 'path'); they have many unusual properties, especially in higher-dimensional spaces, some of which will be discussed in the appendix. For purposes of clarity, their time dependence has been suppressed.

$$\begin{array}{ccccc}
\{\sum f(x,t)\} & \xrightarrow{(8)} & \{\sum f(x,t+1)\} & \xrightarrow{(8)} & \cdots \\
\Big\downarrow\mathscr{H} & & \Big\downarrow\mathscr{H} & & \\
\{\psi(x,t)\} & \xrightarrow{(2)} & \{\psi(x,t+1)\} & \xrightarrow{(2)} & \cdots
\end{array}$$

Figure 2. There exists a homomorphism $\mathscr{H}$ from the space of arc solutions to the space of point solutions that is preserved under their respective time evolutions.

A moment's thought will show that, under the homomorphism, these arc solutions correspond to the point solutions that at $t=0$ are equal to δ_{x,x_0} and at $t=1$ are respectively equal to $\delta_{x,x_0\pm1}$, where $\delta_{x,x'}$ is the Kronecker delta.

Next, let $G_0(x-x_0)$ and $G_1(x-x_1)$ be the point solutions of the wave equation whose initial conditions (at times t_0 and $t_1=t_0+1$) are

$$G_0(x-x_0)=\delta_{x,x_0}\delta_{t,t_0} \qquad G_1(x-x_1)=\delta_{x,x_1}\delta_{t,t_1} \qquad x,x_0,x_1\in\{0,1,\ldots,N-1\}. \quad (11)$$

Again, their dependence on time has been suppressed.

It is clear that any solution of the wave equation may be expressed as a linear combination of the G_0 and G_1. On the subspace of length N mentioned above, the function $G_0(x-x_0)$ and $G_1(x-x_1)$ may be expressed in terms of the arc functions h_+ and h_- as

$$G_0(x-x_0)=\tfrac{1}{2}\sum_{x'=0,2,\ldots}^{N-2}[h_+(x-x_0-x')\,\mathrm{sgn}(x-x_0)-h_-(x-x_0-x')\,\mathrm{sgn}(x-x_0-1)] \quad (12a)$$

and

$$G_1(x-x_1)=\tfrac{1}{2}\sum_{x'=1,3,\ldots}^{N-1}[-h_+(x-x_1-x')\,\mathrm{sgn}(x'-x_1)+h_-(x-x_1-x')\,\mathrm{sgn}(x'-x_1)] \quad (12b)$$

where

$$\mathrm{sgn}(x)=\begin{cases}1 & \text{if } x\geqslant 0\\ -1 & \text{otherwise.}\end{cases}$$

Again, note that if t_0 is even (odd), then the solutions in question are assumed to be non-zero on the even (odd) sublattice; likewise, it is assumed that x_0 is even (odd), while x_1 is odd (even).

As the equations show, these Green functions for the discrete wavefunction may be constructed by interlocking positive and negative hodotic solutions of opposite sign (figure 3). Therefore, the travelling waves completely suffice to specify the solutions of the wave equation.

Note that these Green functions require non-zero flows across the entire space. Therefore, even if the solution one wishes to simulate is initially non-zero only in some localized proper subset of the space, say one that can be covered by a line segment (or in n dimensions, an n-cube), its expression in terms of the Green functions involves non-zero flows in a region extending across the entire space. However, if this (point) solution has the additional property that its discrete integral over space is constant (as

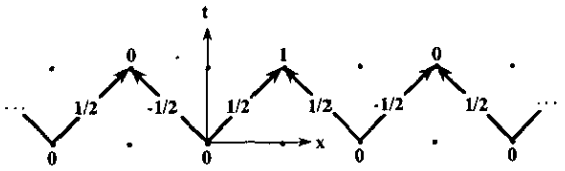

Figure 3. An arrangement of hodotic solutions that yields a Green function $G_1(x-x_1)$ for the discrete wave equation.

(2) implies it would be, if it is the same at any two successive times), then one can find an arrangement of hodotic solutions that is also non-zero only in a localized subset.

2.2. The two-dimensional case

In passing to the two-dimensional case via the present analysis, solutions of the wave equation are again to be expressed as a linear sum of components. Each of these components will likewise be associated with flows along the arcs connecting nearest neighbours. However, since in higher dimensions even a localized wavepacket spreads and deforms, it is to be expected that there will be mixing among these modes of propagation, instead of the trivial translation found in the one-dimensional case.

At any time, let each lattice point be viewed as a kind of 'black box', into which enter and from which exit four amplitudes. The latter are to be determined solely in terms of the former, in a linear fashion. If, as in the one-dimensional case, the flow out from the point (x, y, t) and in to the point $(x+1, y, t+1)$ is denoted as $f^{\text{out}}_{x_+}(x, y, t)$, or equivalently as $f^{\text{in}}_{x_+}(x+1, y, t+1)$, and the flows to other points are analogously denoted, these considerations may be restated as

$$\psi(x, y, t) = \sum_{\sigma} f^{\text{in}}_{\sigma}(x, y, t) = \sum_{\sigma} f^{\text{out}}_{\sigma}(x, y, t) \tag{13}$$

and

$$f^{\text{out}}_{\sigma} = \sum_{\sigma'} c^{\text{in}}_{\sigma,\sigma'} f^{\text{in}}_{\sigma'} \tag{14}$$

where $\sigma, \sigma' \in \{x_+, x_-, y_+, y_-\}$, and where the coefficients of the matrix $c^{\text{in}}_{\sigma,\sigma'}$ are to be determined.

Consider the solution of the wave equation corresponding to a non-zero flow in only one arc, say $f^{\text{in}}_{x_+}(0, 0, 0) = 1$, with all other incoming flows to all other points being zero; i.e. a two-dimensional hodotic solution. As in the one-dimensional case, the boundary conditions for the corresponding point solution are $\psi(x, y, 0) = \delta_{x,0}\delta_{y,0}$, and $\psi(x, y, 1) = \delta_{x,1}\delta_{y,0}$.

Since the wave amplitude is thus specified on the entire space at two successive times, one can iterate these boundary conditions according to the wave equation (figure 4), and thus uniquely determine the first column of the matrix $c^{\text{in}}_{\sigma,\sigma'}$. Repeating this

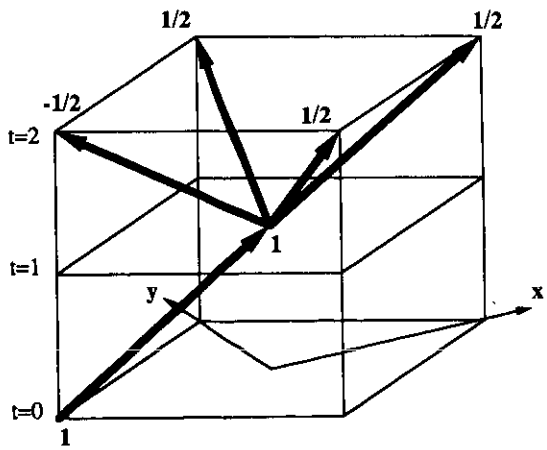

Figure 4. The x_+ hodotic solution, at $t = 0$, 1 and 2.

analysis for the hodotic solutions of the remaining three directions completes the specification of the matrix, so that

$$\begin{pmatrix} f_{x+} \\ f_{x-} \\ f_{y+} \\ f_{y-} \end{pmatrix}^{\text{out}} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \end{bmatrix}\begin{pmatrix} f_{x+} \\ f_{x-} \\ f_{y+} \\ f_{y-} \end{pmatrix}^{\text{in}}. \tag{15}$$

By using the time invariance properties of the wave equation, one also could have just as easily determined the coefficients of the inverse of the above matrix, $c^{\text{out}}_{\sigma,\sigma'}$, and would have found them to be identical. It is then a matter of simple algebra to show that the sum of the ingoing or outgoing flows at any point does indeed obey the discrete wave equation.

Note that the sum of the coefficients along any column of the matrix is one, so that the sum of flows entering a point at any time is equal to the sum of the exiting flows. This is also obvious from (13). Furthermore, the matrix is unitary. Therefore, the sum of the squares of the entering (and thus the exiting) flows is conserved. Thus the evolution of such a system can be viewed as a network flow of a conserved quantity.

2.3. Higher-dimensional cases

In higher dimensions, the hodotic solutions $h_{i\pm}(\boldsymbol{x}-\boldsymbol{x}_0, t)$ are defined (in terms of their point values on the lattice) by the initial conditions

$$h_{i\pm}(\boldsymbol{x}-\boldsymbol{x}_0, t) = \begin{cases} \delta_{\boldsymbol{x},\boldsymbol{x}_0} & \text{for } t = t_0 \\ \delta_{\boldsymbol{x},\boldsymbol{x}_0\pm\boldsymbol{e}_i} & \text{for } t = t_0+1 \end{cases} \tag{16}$$

where the jth component of $\boldsymbol{e}_i$ has the value δ_{ij}. In terms of their arc amplitudes, the hodotic solutions are initially equal to unity on the $x_{i\pm}$ arc leading out of the point $\boldsymbol{x}_0$, and zero on all the others.

The method of using (13) and (14), and the evolution of the hodotic solutions in order to determine the coefficients of the transition matrix, can be readily generalized to higher dimensions. For example, in three dimensions, the transition matrix is determined to be

$$\begin{pmatrix} f_{x+} \\ f_{x-} \\ f_{y+} \\ f_{y-} \\ f_{z+} \\ f_{z-} \end{pmatrix}^{\text{out}} = \frac{1}{3}\begin{bmatrix} 1 & -2 & 1 & 1 & 1 & 1 \\ -2 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -2 & 1 & 1 \\ 1 & 1 & -2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & -2 \\ 1 & 1 & 1 & 1 & -2 & 1 \end{bmatrix}\begin{pmatrix} f_{x+} \\ f_{x-} \\ f_{y+} \\ f_{y-} \\ f_{z+} \\ f_{z-} \end{pmatrix}^{\text{in}}. \tag{17}$$

In the general n-dimensional case, the coefficients of the matrix in any row or column will be $1/n$, except for the coefficients connecting oppositely directed flows (e.g. c_{x_+,x_-}), which will be $-(n-1)/n$.

In each case, the number of travelling waves is determined by the number of adjacent neighbours of the points of the lattice, and the evolution of a given wave may be determined by the study of the corresponding hodotic solutions.

The considerations of completeness may be directly generalized to higher dimensions merely by imbedding the one-dimensional expressions for the Green functions (12) along any of the axes. This is arguably the simplest way to express the Green functions, although in higher dimensions more elaborate expressions are possible (figure 5), since the overcompleteness of the hodotic solutions is more extensive.

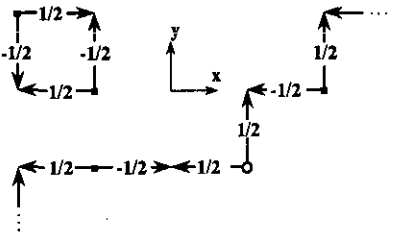

Figure 5. A two-dimensional non-collinear arrangement of hodotic solutions that yields a Green function for the discrete wave equation. Nearest neighbour points are displaced by one unit of time. The circle marks the support of the point solution.

It is possible to formally extend the notions of travelling waves to non-orthonormal lattices, generalizing the discrete wave equation to hexagonal, tetragonal and, indeed, to any set of points where each point j has some privileged subset $\mathcal{N}(j)$ of generalized nearest neighbours; the 'wave equation' then becomes

$$\psi(j, t+1)+\psi(j, t-1)=\frac{2}{|\mathcal{N}(j)|}\sum_{j'\in\mathcal{N}(j)}\psi(j', t) \tag{18}$$

where $|\mathcal{N}(j)|$ is the number of nearest neighbours—not necessarily constant throughout the space—corresponding to the point j. Again, each pair of neighbours will induce a flow, with an interaction among flows that may be determined by studying the behaviour of the associated hodotic solutions. As before, there will be a conservation of the sum of the flows, and also of the sum of their squares, since the transition matrix is in all cases unitary. Of course, how well, if at all, the resultant system mimics the continuum wave equation in some limit depends on the particular lattice employed and, if definable, the associated dispersion relation.

3. Path summations

The same Monte Carlo (and related) methods that are commonly used to simulate diffusion phenomena [2–7] may be extended to the simulation of the wave equation, using the results of the previous section. Such an approach is to be distinguished from the related method of simulating lattice wave solutions by density variations of lattice gases (similar to those obtained in the study of Ising systems [8]). The validity of such an approach may be examined by way of the associated (linearized) Boltzmann equation [2–4, 9–12]. It is also possible to obtain a cellular automaton model of the wave equation directly, without recourse to path summations and diffusion phenomena [13], though the present approach is more extensible to the study of Klein-Gordon and related equations relevant to physics.

We briefly recapitulate some results of the random walk as applied to the diffusion equation, in order to emphasize the similarity of that formalism to the present one.

The exposition given here is presented in such a way as to anticipate and facilitate its subsequent application the wave equation.

Consider a discrete dynamical system consisting of particles executing random walks on an orthonormal n-dimensional lattice. At any step in the (discrete) evolution of the system, particles are to be found at some lattice point, and they move to a randomly chosen nearest-neighbouring point in the subsequent time step.

The probability that a particle initially at the lattice point x_0 will at time T be found at x may then be written as

$$P(x;\, T|x_0;\, 0) = \sum_I W(I) \tag{19}$$

where I is an indexing of the set of $(2n)^T$ lattice *paths* originating at x_0, ending at x, and containing T steps. If one assumes that the probability $p_{j\pm}$ for taking a step along a given direction $x_{j\pm}$ is everywhere constant, then the weighting factor $W(I)$ has the value

$$W(I) = (p_{1_+})^{r_{1_+}(I)}(p_{1_-})^{r_{1_-}(I)} \ldots (p_{n_+})^{r_{n_+}(I)}(p_{n_-})^{r_{n_-}(I)} \tag{20}$$

where $r_{i_\pm}(I)$ is the number of steps along the $x_{i_\pm}$ direction that are found in the Ith path.

In the case where all the $p_{i_\pm}$ are equal, the right-hand side of (19) can be shown to converge in the continuum limit to the kernel for the n-dimensional diffusion equation [14].

3.1. Lattice ensembles

One can use (19) to give diffusion phenomena is statistical implementation. By using an *ensemble* of appropriately initialized lattices (with the initialization procedure to be discussed below), one can use the distribution of particles on these lattices to simulate a solution of the diffusion equation $f(x, t)$. Like any formalism, $f(x, t)$ is assumed to be approximately constant over the length of the lattice spacing (and over any time interval the length of the fundamental time increment). It is also assumed to be bounded, normalized so that its maximum value is initially unity, and, for now, positive.

Let the number of lattices in the ensemble be some very large number M. Define $n_j(x)$ to be the number of particles at the point x at time t in the jth lattice, where j ranges from 1 to M, and where the time dependence will customarily be understood.

To say that at time t the *statistical amplitude* at x is $f(x, t)$, is to say that

$$f(x, t) = \frac{1}{M}\sum_{j=1}^{M} n_j(x) \tag{21}$$

regardless of how the occupation numbers vary from lattice to lattice. Likewise, it will be said that an ensemble of lattices *statistically simulates* the function $f(x, t)$ if the above relation holds. (In any practical implementation, the above equals sign must be interpreted to mean 'approximately equals, to the desired degree of accuracy'.)

3.2. Initialization

Next, consider how to initialize the ensemble of lattices corresponding to $f(x, 0)$, beginning with the following definition. Performing an action A '*with a probability p*' is defined as first obtaining a random number ζ, uniformly distributed between 0 and

1. If $\zeta \leqslant p$, then action A is performed; otherwise, it is not. The random numbers ζ obtained from multiple repetitions of such actions are assumed tt be statistically independent.

Each lattice of the ensemble is to be initialized independently of the others. At the point x of, say, the jth lattice, one places a particle there 'with a probability of $f(x, 0)$'. One then repeats this procedure on every other point x of the lattice.

Every lattice in the ensemble, with j ranging from 1 to M, is initialized in this same way. This of course means that in general, there will be more than one particle per lattice. (Indeed, one could in this case have chosen simply to place the particles from all the M lattices onto one single lattice but, again, the exposition given here is made in such a way as to facilitate its application to the wave equation.)

Let us suppose that at time T an ensemble of lattices statistically simulates $f(x, T)$. In each subsequent time step, let each particle move to one of its $2n$ nearest neighbours, the choice being made randomly for each particle. Assume there is no restriction on the number of particles that can be found at given point on any lattice.

By using basic probability theory, one may use (19) to show that an ensemble of lattices initialized according to the preceding procedure will continue to statistically simulate $f(x, t)$ at each subsequent time step. In order to simulate phenomena lasting T time steps, a number of lattices on the order of $(2n)^T M_0$ will be required in general, where M_0 is the number of lattices required to initially statistically simulate a given solution to the desired degree of accuracy.

3.3. Extensions to complex solutions

There is, of course, nothing about the diffusion equation that requires the solutions to be real. Suppose that each lattice particle is endowed with an additional degree of freedom corresponding to a discrete phase factor, having one of the four possible values of 1, +i, −1 and −i; particles in these respective phases will, respectively, be referred to as being positive, posimaginary, negative and negimaginary.

Assuming that at the lattice point x of the jth lattice there are a positive particles, b posimaginary particles, c negative particles and d negimaginary particles, let the definition of the occupation number $n_j(x)$ be modified so that

$$n_j(x) = (a - c) + \mathrm{i}(b - d)$$

therefore, this 'occupation number' is now in general a complex integer.

Moreover, it is assumed that oppositely phased particles (positive versus negative, posimaginary versus negimaginary) found at the same lattice point at any time annihilate each other, leaving behind particles of at most two phases.

One can then use ensembles of lattice particles to statistically simulate complex solutions as well. To initialize the ensemble of lattices to correspond to the solution $f(x, t)$, where $f(x, t)$ may now be complex, one first defines the four positive functions

$$\begin{aligned} f_{\mathrm{Re}\pm}(x, t) &= \pm \frac{1 \pm \mathrm{sgn}(f(x, t))}{2} \mathrm{Re}(f(x, t)) \\ f_{\mathrm{Im}\pm}(x, t) &= \pm \frac{1 \pm \mathrm{sgn}(f(x, t))}{2\mathrm{i}} \mathrm{Im}(f(x, t)) \end{aligned} \tag{22}$$

after which one simply initializes the ensemble according to each of the four functions simultaneously, in each case using the correspondingly phased particles; positive

particles for $f_{\mathrm{Re}+}$, posimaginary particles for $f_{\mathrm{Im}+}$ and so on. (It is assumed that $f(x, 0)$ has been normalized so that none of the maxima of the above four functions exceed unity.) Therefore, at the end of this initialization every lattice will in general contain all four kinds of particles, though at any given point on the lattice there will be particles of at most two phases.

3.4. Applications to the wave equation

To statistically simulate a solution of the wave equation, one must first expand the point solution (which, as stated previously, is specified by its values on the *points* of the lattice at two subsequent time steps) into an arc solution, by way of the Green functions (12). It is convenient to normalize the solutions so that the (discrete) integral of the squares of the arc ampltudes is unity, i.e.

$$\sum_{\{x,\sigma\}} |\psi_\sigma(x, t)|^2 = 1 \tag{23}$$

given that this quantity is conserved. (The summation indices denote that the summation is taken over all the nearest-neighbour arcs of the lattice.)

Then, one initializes the lattices as in the case of the diffusion equation, except that instead of placing particles at *point* x—with a probability and phase dependent on the amplitude at that point—one now places particles in the *arcs* leading out from x, in likewise accordance with the amplitude at those arcs. A particle in the σ arc of the point x will then be assumed to execute a step in the σ direction in between the times $t=0$ and $t=1$, where the $2n$ possible values of σ again represent positive or negative directions along the axes of the lattice.

Just as in the above section dealing with travelling wave solutions, the lattice points should here again be viewed as 'black boxes', into which particles enter, and out of which other particles are generated. Explicitly, the dynamics is such that a particle taking a step terminating at some lattice point will cause that lattice point to *generate* particles in all the outgoing arcs (with probability distributions to be discussed below). In other words, the particle numbers will no longer be constant, so that it is only be averaging that one recovers the conservation of amplitude and its square that is implied by unitarity of the transition matrices.

The particles in the outgoing arcs will in the subsequent time step travel along those arcs to the corresponding nearest neighbouring lattice point, where the generating process will be repeated. (The parent particle is assumed to annihilate after reaching its destination lattice point.)

Consider next the transition matrices for the n-dimensional generalization of (14). Let $|c_{\sigma,\sigma'}|$ designate the *probability* that a particle coming into a lattice point along the σ' arc will produce an outgoing particle in the σ arc. If $c_{\sigma,\sigma'}$ is positive, then the outgoing particles will have the same phase factor, or sign, as the incoming particles. If it is negative, the outgoing particles will have the *opposite* phase of the incoming particles.

For example, in the two-dimensional case (figure 4), a positive particle coming into a lattice point along the x_+ arc will produce 'with a probability $\frac{1}{2}$' a positive particle in the y_+ (or y_-) arc. It will also produce 'with a probability $\frac{1}{2}$' a *negative* particle in the x_- arc, this change of sign being mandated by the fact that the coefficient c_{x_-,x_+} is negative. The generalization to other dimensions is straightforward.

It is assumed that a parent particle generates output particles in the outgoing arc σ completely independently of the particles it produces in any other arc σ'. Moreover, if there is more than one particle entering a lattice point, the particles emitted because of incoming particle A are generated independently of the particles emitted because of some other incoming particle B. (Once the particles are generated, it is again assumed that oppositvely phased particles found simultaneously in any *arc* annihilate each other.)

An ensemble of lattices that initially simulates the wave equation will then continue to do so in subsequent time steps, as may be shown by the same calculation as in the case of the diffusion equation. If an ensemble of size M is initialized to simulate the wave solution $\psi(x, 0)$ then, in order to obtain the (point) solution at any other time T, one allows the ensemble of lattices to evolve for T time steps and then obtain the quantity

$$(1/M)\sum_{j=1}^{M} n_j(x, T) \tag{24}$$

where $n_j(x, T)$ now refers to the sum of the occupation numbers for the arcs leading into the point x (at time T in the jth lattice). Again, simulating phenomena lasting T time steps to within an initially prescribed accuracy will require a number of lattices on the order of $(2n)^T M_0$, where M_0 is as before. Note that even though the wave equations are time-reversal invariant, the dynamics used in their simulation are asymmetrical with respect to time reversal, and it is only by averaging that the symmetry is recovered.

The relationship between amplitudes and paths on the lattice that exists in the case of the diffusion equation may be retained in the present case. A particle coming into x along the σ arc that produces an outgoing particle in the σ' arc still specifies a path increment; any particle generated in another outgoing arc likewise represents the increment of yet another path. Given that in the general n-dimensional case the coefficients of the transition matrices are $1/n$ and $(1-n)/n$, one may heuristically say that the (generalized) probability of a particle making a path increment along an incoming arc σ to an outgoing arc σ' is $1/n$, unless the two arcs are oppositely oriented (implying that the associated path increment is a 'reverse step', or *reversal*), in which case it is $(1-n)/n$.

Relation (19) has an analogue in the case of the wave equation, in that the hodotic solutions (and, given the completeness thereof, any solution of the wave equation) can similarly be expressed in terms of summations over paths. In fact, in accordance with the considerations of the previous paragraph, the n-dimensional hodotic solution has the likewise expansion

$$h_\sigma(x-x_0, T)=\sum_{I_\sigma} W(I_\sigma) \tag{25}$$

where the $2n$ possible values of σ again represent positive or negative directions along the lattice, and where I_σ is an indexing of lattice paths of length T whose initial step is along the σ-direction. In the present case,

$$W(I_\sigma)=(1/n)^T(1-n)^{R(I_\sigma)} \tag{26}$$

where $R(I_\sigma)$ is the number of reversals in the path I_σ; for the one-dimensional case $n=1$, zero to the zeroth power is defined to be one.

4. The Klein–Gordon equation

The above results for the wave equation can be extended to the computation of the Klein-Gordon equation. On an m-dimensional Euclidean space, the Klein-Gordon equation takes the form

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi = \left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\ldots+\frac{\partial^2}{\partial x_m^2}\right)\psi+\eta^2\psi \tag{27}$$

where η is a real parameter. (Note again that the above equation is in Cartesian space, so that the partial second derivatives have their usual definitions.) The Fourier frequencies of this equation obey a dispersion relation of the form

$$\omega^2c^2 = k_1^2+k_2^2+\ldots+k_m^2+\eta^2.$$

4.1. Specification of dimensions

Consider a toroidal space of n dimensions, all except m of which are of length two (so that along such a dimension $\psi(\ldots, x_i+1, \ldots)=\psi(\ldots, x_i-1, \ldots)$). The lattice is still effectively m dimensional (i.e. their macroscopic behaviour simulates a wave equation whose speed of light is $\sqrt{1/n}$), but there are also extra degrees of freedom to consider.

First suppose that there is only one thin dimension, with $n-m=1$. In comparison to the standard (i.e. $n=m$) m-dimensional systems, there now exist another set of spatial frequencies, κ_{x_n}, that are required to specify the system, which correspond to motions along the thin dimension. However, since the index x_n only takes the values '0' or '1' (or by an appropriate change of variables '$\pm\frac{1}{2}$'), it could be viewed as merely a specifier between two different 'components' of the solutions.

The dispersion relation of this system would then be of the form

$$\sin^2\frac{\omega}{2} = \frac{1}{n}\left[\sin^2\frac{\kappa_{x_1}}{2}+\ldots+\sin^2\frac{\kappa_{x_m}}{2}+\sin^2\left(\frac{\pm\pi}{2}\right)\right] \tag{28}$$

(the last term of course being equal to one). For extremely small lattice spacings, this expression would become, in analogy to the discrete wave equation dispersion relation described above,

$$\omega^2c^2 = k_1^2+\ldots+k_m^2+\left(\frac{\alpha}{d}\right)^2 \tag{29}$$

where α is a constant of the order of one, and d is the length of the lattice spacing. This is precisely of the form of the Klein-Gordon dispersion relation, with d/α acting as the Compton wavelength of the particle whose wavefunction obeys the Klein-Gordon equation.

This approach therefore is loosely reminiscent of Kaluza–Klein models, insofar that mass is the manifestation of the topology of the space corresponding to motion along a 'matter' dimension. Of course, this system is only useful for situations involving spatial frequencies, i.e. momenta, much smaller than d^{-1}, so that only non-relativistic phenomena can be accurately simulated.

Suppose, however, that m is the typical 2 or 3, corresponding to two- and three-dimensional cases, and that n is large, even astronomically so. Such a system has an evolution insufferably hard to compute (the speed of light is unbearably 'slow', i.e. a

relatively large amount of computation is necessary to execute a given time interval of the simulation). Even so, the above considerations for the case $n = m + 1$ still hold, except that the mass term in the dispersion relation, α, would be multiplied by a factor of n, thereby increasing the spectral range in which the approximation to the continuous case is good. Also, the approximately n momenta corresponding to motion along the thin dimensions can likewise be further exploited in ways that depend on the internal structure of the particles one wishes to simulate.

Such a model is also reminiscent of the one-dimensional Klein-Gordon equation considered by Feynman [15, 16], which he labels a Dirac equation. Although the system under consideration here has the advantage of being applicable in any dimension (Feynman was unable to extend his version beyond one dimension [16, 17]), both versions are limited by the fact that the lattice spacing may no longer be taken to zero, but has a fixed length proportional to the Compton wavelength of the particle under consideration. Even if one could disregard the problems of accommodating such a model with the continuity of space that has been observed at all experimentally accessible spatial scales, it would still be more desirable to *a priori* require the model to accommodate an arbitrarily small lattice spacing so that, in principle, the continuum could then be replicated to arbitrary precision. The remainder of this section presents such a model. However, it should be noted that the previous approach is useful in that it demonstrates how the topology of the lattice is itself a set of parameters, which can be used to alter the dynamics in accordance with the desired equations of motion.

4.2. The modulation method

Another way of obtaining the Klein-Gordon equation from the wave equation is to first double the number of neighbouring arcs. For example, in two dimensions, a point then has two arcs connecting any of its nearest neighbours: two arcs in the x_+ direction, two in the y_+ direction and so on. Alternatively, one may suppose that the associated tokens have yet another degree of freedom and can be, in either, say, a 'top' or 'bottom' state. Each one of the doubled number of arcs will, just as before, lead to separate flows, which will be respectively denoted by f_{x_+} and g_{x_+}, etc. The equation of motion for the travelling waves is then given by the system of equations

$$\begin{pmatrix} f_{x_+} \\ f_{x_-} \\ f_{y_+} \\ f_{y_-} \\ g_{x_+} \\ g_{x_-} \\ g_{y_+} \\ g_{y_-} \end{pmatrix}^{\text{out}} = \frac{1}{4} \begin{bmatrix} 1 & -3 & 1 & 1 & 1 & 1 & 1 & 1 \\ -3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -3 & 1 & 1 & 1 & 1 \\ 1 & 1 & -3 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & -3 & 1 & 1 \\ 1 & 1 & 1 & 1 & -3 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & -3 \\ 1 & 1 & 1 & 1 & 1 & 1 & -3 & 1 \end{bmatrix} \begin{pmatrix} f_{x_+} \\ f_{x_-} \\ f_{y_+} \\ f_{y_-} \\ g_{x_+} \\ g_{x_-} \\ g_{y_+} \\ g_{y_-} \end{pmatrix}^{\text{in}} . \tag{30}$$

The system is still, in effect, two-dimensional. Indeed, one sees that by adding the respective 'top' and 'bottom' coefficients of the matrix and treating the quantity $f_\sigma + g_\sigma$ as one flow, the original two-dimensional system of equations (15) is recovered.

Next, divide the above transition matrix into four equal quadrants, and multiply the entries in the two diagonal quadrants by $e^{i\alpha}$ and the ones in the off-diagonal quadrants by $e^{-i\alpha}$, where α is some small (real) constant. How this modulation can be imposed by way of a discrete particle-like process is discussed below.

The equation this system satisfies (as may be shown by direct computation) is then identical to (2) except that the speed of light is multiplied by a factor of $\cos\alpha$. The resultant dispersion relation for this system is then

$$\cos\omega = \frac{\cos\alpha}{2}(\cos\kappa_x + \cos\kappa_y). \tag{31}$$

Note again that, as in the case of the wave equations, the frequencies are always real. Expanding the above equation in a power series shows that, for a small lattice spacing, this system satisfies (for sufficiently small α) a dispersion relation of the form

$$\omega^2 = \frac{1}{2}\left[k_x^2 + k_y^2 + \left(\frac{\alpha}{d}\right)^2\right] \tag{32}$$

where the k_x and k_y have their usual continuum definitions, and where d is now a length on the order of the lattice spacing. This, again, is precisely the form of the dispersion relation of the Klein–Gordon equation. Note that by negating α, another system satisfying the same dispersion relation is obtained. This approach can immediately be extended to any number of dimensions and has the additional advantage over the previous one in that the extra components that have been introduced do not change the value of the speed of light from $1/\sqrt{n}$, where n is the corresponding effective dimension.

4.2.1. Statistical modulation

The lattice interaction whereby an incoming flow is modulated by a factor of $e^{i\alpha}$ *from* what it would have been in the case of the wave equation may be implemented via a particle approach, just as previously the flows associated with the wave equation were interpreted in terms of the statistical motions of discrete particles. It will next be shown that a gas-like population of Poissonly distributed background particles can be used to effect (statistically) the desired modulation.

Consider again the lattice in which the Klein–Gordon equation is to be implemented. Aside from the tokens executing wave-like motion, let there also be a number of particles comprising what will be called the *moderator* gas. The dynamics of these particles as they move from point to point along the arcs is arbitrary, except that it is assumed that they have at all times a Poisson distribution. That is, at each arc, there is a probability

$$P(\alpha; k) = \frac{\alpha^k}{k!}e^{-\alpha} \tag{33}$$

of finding exactly k particles. Therefore, if the number of points in the lattice is N^n, the expected number of moderator tokens, assuming $2n$ arcs per point, is $2n\alpha N^n$.

Suppose next that the presence of one moderator token in an incoming arc multiplies the outgoing distribution by $(1+i\varepsilon)$, where ε is a positive number which shall be assumed to be less than or equal to one. That is, whereas in the absence of any moderator tokens the presence of a wave token of phase ϕ produces outgoing wave tokens of the same phase with a probability of $\frac{1}{4}$, now there will also be produced, with a probability of $\varepsilon/4$, a wave token of phase $i\phi$. For the reverse arc, where the probability associated with the outgoing tokens is $\frac{3}{4}$ and the phase of the outgoing tokens is $-\phi$, the tokens induced by the moderator particles are likewise produced with a probability of $3\varepsilon/4$ and a phase of $-i\phi$. (As always, any resultant tokens of

opposite phases annihilate each other.) In the absence of any moderator tokens, there is no extra ε-interaction.

Thus, the moderator tokens act to multiply the number of wave tokens. In a given arc, in the presence of one moderator token, the input of one wave token leads to an expected number of $(1+i\varepsilon)$ tokens. As additional moderator token will act on the output tokens of the first one in the same way, resulting in an expected number of $(1+i\varepsilon)^2$ tokens.

Therefore, if the distribution of moderator tokens is given by (33), the net expected number of output tokens is

$$e^{-\alpha}+(1+i\varepsilon)\frac{\alpha}{1!}e^{-\alpha}+(1+i\varepsilon)^2\frac{\alpha^2}{2!}e^{-\alpha}\ldots=e^{i\varepsilon\alpha}.$$

Thus, a discrete process is sufficient to modulate the statistical flows of tokens by an arbitrary real parameter. The moderator distribution thus serves here a function similar to that of the Higgs fields of quantum field theories, insofar as it induces a mass. Note that if the extra token which the moderators produced are of a phase $-i$ times the phase of the initial token, making ε, in effect, a negative number, then the corresponding modulation factor likewise has a negative phase. The remainder of this section shows that moderator particles that are themselves multiply phased can also be used to induce a modulation factor.

Suppose next that there are two types of moderator tokens, each with a Poisson distribution parametrized by α and α', respectively, and both of which operate on the wave tokens of phase ϕ to produce an additional token of phase $i\phi$, with the probabilities ε and ε'. The modulation is then $e^{i\varepsilon\alpha+i\varepsilon'\alpha'}$.

Next consider a situation in which the two types of moderator tokens, each initially with a Poisson distribution parametrized by α_+ and α_-, at each arc of the lattice, are given the respective opposite phases of $+1$ and -1. That is, the modertor tokens are themselves given phases, with all the α_+ tokens having a positive phase, and all the α_- tokens having a negative one. (Note that imaginary phases are not included.) Each positive and negative tokens multiplies the outgoing distribution of wavefunction tokens by a factor of $(1+i\varepsilon)$ and $(1-i\varepsilon)$, respectively. Where moderator tokens of both types are found in any arc of the lattice, they again will be made to annihilate, so that only one type of token remains.

Thus, in considering both types of tokens at once, one can say that at any arc of the lattice there is a certain probability of finding any number k of tokens, where k may be negative as well as positive. Finding a negative number of tokens simply means finding a positive number of tokens of the second type.

The probability of finding any number k is then given by

$$P(\alpha_+,\alpha_-;k)=\begin{cases}\sum_{m=0}^{\infty}P(\alpha_+;k+m)P(\alpha_-;m) & k\geqslant 0\\ \sum_{m=0}^{\infty}P(\alpha_+;m)P(\alpha_-;m-k) & k<0.\end{cases}\tag{34}$$

Writing out the individual Poisson distributions, one obtains

$$\begin{aligned}P(\alpha_+,\alpha_-;k)&=e^{-\alpha_+-\alpha_-}\sum_{m=0}^{\infty}\frac{\alpha_+^{|k|+m}\alpha_-^{m}}{(|k|+m)!\,m!}\\&=e^{-\alpha_+-\alpha_-}\left(\frac{\alpha_+}{\alpha_-}\right)^{k/2}I_{|k|}(\sqrt{4\alpha_+\alpha_-}) \qquad k=\ldots,-1,0,1,2,\ldots\end{aligned}\tag{34}$$

where $I_k(x)$ is the modified Bessel function of the first kind, of order k, which satisfies the differential equation

$$x^2 I''(x) + xI'(x) = (x^2 + k^2) I(x).$$

The modulation factor arising from such a distribution of tokens is proportional to $(\alpha_+ - \alpha_-)$.

In the limit of infinitely long times, the same kind of modulation can be produced by a moderator with a binomial instead of a Poisson distribution, because of the similarity of the two when the appropriate limits are taken. (By a binomial distribution, what is meant is that there is a probability p of finding one moderator token at any arc, and a probability $(1-p)$ of finding none.) The ability to statistically modulate quantities by an arbitrary phaseshift is useful not only for the simulation of the Klein–Gordon equation, but for the analogous equation for particles subject to electromagnetic or other potentials that themselves satisfy wave equations. The potential amplitude can, just as the quantum wavefunctions and the moderator, be given statistically with discrete tokens, which will operate on the quantum wavefunctions just as the moderator tokens do.

5. Conclusion

There have been numerous efforts to relate wave phenomena to stochastic phenomena; in physics these attempts were often directed toward illuminating quantum mechanics [18–20]. Some of these efforts were motivated by the similarity of the Schrödinger equation to the diffusion equation [21–22], despite the consequent difficulty of making the resulting formalism relativistically covariant. Others involve analytically continuing the time variable into the complex domain, thereby rendering the propagators amenable to Monte Carlo computations [23]. The present formalism is a departure from these previous approaches in that it starts directly with relativistically covariant systems governed by the standard (i.e. the un-Euclideanized) spacetime metric. The Schrödinger equation is then obtainable as the low-velocity limit of the Klein–Gordon equation.

However, all of the concepts underlying the present work have been studied before. In particular, the similarity to a related model introduced by Feynman has already been noted. Also, S Gudder has studied the constraints an abstract lattice system or graph must satisfy if it is to simulate quantum mechanics (and its associated wave equations [24–25]. Still others have argued that lattices, discrete networks, and other continuum-violating paradigms have an importance beyond their utility as computational or analytical aids, given that continuity as experienced in the macroscopic world need not apply at sufficiently microscopic scales [26–32].

Finally, it should be noted that the present formalism is itself particularly well suited to the study of single and multiparticle quantum mechanics, in that it may be used to compute the square of the wavefunction directly, as opposed to obtaining the square only after having computed the wavefunction itself. Just as the present approach allows one to obtain the solution of a lattice wave equation at some point by counting the (properly weighted) lattice particles, there exists a quantity likewise defined on a lattice (also taking on discrete values) which when summed over an ensemble of similarly prepared systems is proportional to the square of a wavefunction of a multiparticle wavefunction—all in a three-dimensional configuration space governed by a dynamics that is local within that space and, as mentioned before, all taking place

in un-Euclideanized time [33]. Although the expected value of this quantity is therefore non-negative, the quantity itself may assume negative values at sufficiently microscopic regimes, though the probability of this occurring becomes vanishingly small as the ensemble of systems under consideration becomes large. This unphysical feature provides the loophole that allows the constraints of Bell's inequality to be circumvented.

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Appendix. Hodotic solutions

Some unusual properties of the hodotic solutions of the wave equation are easily established by resorting to summations over paths. As before, let $\Sigma_x f(x, t)$, be the sum over all space of some function $f(x, t)$ at some instant of time. That is, it is a sum over the *point* values of a given solution.

It is easy to show from the equations of motion that the hodotic solutions have the property that $\Sigma_x h_\sigma(x, t)$ is equal to one for all time where, as before,

$$\sigma \in \{x_{1_+}, x_{1_-}, x_{2_+}, \ldots, x_{n_-}\}.$$

What is much more remarkable is that on any n-dimensional torus whose lengths along any of the coordinate axes are the same, the quantity $\Sigma_x h_\sigma^2(x, t)$ is also equal to one for all time. That is, without regard to the travelling wave *arc* solutions, the integral of the square of the *point* solution is conserved. (This property does not hold, in general, for an arbitrary sum of the hodotic solutions.) The proof of this conservation is somewhat tedious, but it will be outlined briefly, using the two-dimensional case as an example.

At noted above, $h_{x_+}(x, y, T)$ may be written as a sum over all walks of length T that start the origin, go immediately afterwards to the point $(1, 0)$, then on to (x, y). By the same argument, $h_{x_+}^2(x, y, T)$ may also be written as a sum of terms, each of which now represents a *pair* of such paths. Explicitly,

$$h_{x_+}^2(x, y, T) = \sum_{I_{x_+}, I'_{x_+}} \left(\tfrac{1}{2}\right)^{2T} (-1)^{R(I_{x_+}) + R(I'_{x_+})} \tag{36}$$

where I'_{x_+} is an indexing of paths indentical to I_{x_+}, and the rest of the notation is again the same as for (25) and (26).

Let $h_{x_+}^\sigma(x, y, T)$ represent the contribution to $h_{x_+}(x, y, T)$ of paths whose Tth (i.e. whose final) step is along the x_+ direction, so that

$$h_{x_+}(x, y, T) = \sum_\sigma h_{x_+}^\sigma(x, y, T). \tag{37}$$

Then,

$$\sum_x (h_{x_+})^2 = \sum_x \sum_\sigma (h_{x_+}^\sigma)^2 + \sum_x \sum_{\sigma \neq \sigma'} h_{x_+}^\sigma h_{x_+}^{\sigma'} \tag{38}$$

where the functional dependence on x, y, and T has been suppressed. Now the first (double) sum on the right is equal to one. This is because it represents the discrete integral of the sum of the squares of the travelling wave components for this solution of the wave equation. This quantity, due to the unitarity of the transition matrix, is always conserved (and therefore equal to its initial value of one). Therefore, it remains only to show that the second sum on the right vanishes.

First suppose that the two-dimensional torus under consideration is of infinite length along either axis. Unlike the first sum on the right-hand side, the second contains no terms representing pairs of walks that are identical. This is because, if the two paths corresponding to such a term were identical, they would obviously have their Tth step along the same direction and therefore would belong in the first sum. It is easy to see graphically that any such pair of non-identical paths may be related in a one-to-one way to another pair of paths (belonging to the same summation) whose cumulative number of reversals differs by one (figure 6). Since the value of the contribution of any pair of paths depends only on the number of reversals they contain, the contributions to the summation from these two sets of paths therefore cancel one another (figure 6). (In fact, one can extend this argument to show that (36) would be unchanged if the right-hand side summands were multiplied by $\delta_{l_{x_+},l'_{x_+}}$.)

The same result can be obtained without resorting to graphical methods by representing each path as a string of T choices from the set of possible directions, and then showing that the contributions from certain classes of strings cancel.

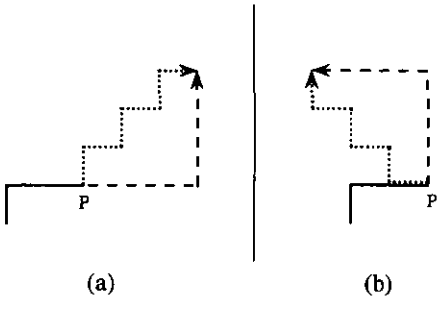

Figure 6. Two paths (a) that are identical except for their final m steps, where $m \neq 0$, but arrive at the same place at $t = T$, may by a symmetry operation be related to another pair of paths (b) such that the total number of path reversals in the latter pair differs by one, so that the net contribution of these two pairs is zero.

To extend these results to a torus, it is also necessary to consider terms in the sum representing *wrapping pairs*, i.e. pairs of paths that when connected end to end describe a loop that is not homotopic to a point, but instead wraps around the torus along some direction. (The reference to homotopy is, of course, made with regard to the continuous paths that can be obtained from the discrete ones by imbedding the lattice in a Euclidean space, and by likewise transforming any step between two lattice points into a continuous path along the straight line segment joining those points.)

If the two axes of the torus are of equal length, then the contributions from the pairs of paths that wrap around the torus along one direction may be seen to cancel the contributions of those that wrap in the perpendicular direction. However, suppose the lengths of the lattice along the two axes are unequal. Then, for sufficiently large T, there will be a pair of paths wrapping around the shorter length of the torus whose

contribution to the integral will not be cancelled by a pair of paths wrapping along the other axis, because the latter axis is too long to traverse in T time steps (figure 7).

In $n>2$ dimensions, the proof is more tedious, because the coefficients in the transition matrix no longer have the same modulus. Let a pair of non-identical paths be called reversal diverging or *trans*versal diverging, respectively, according to whether or not the two paths diverge by having one of the paths execute a reverse step (figure 8). (Remember, all of the paths under consideration already have an identical first step.) The contribution of a pair of reversal-diverging paths must then be added to the contributions of $(n-1)$ pairs of transversal-diverging paths in order to obtain a cancellation. In extending the proof to an n-dimensional torus, one will similarly have to take special notice of the wrapping pairs, which again may be classed according to whether they are reversal diverging or transversal diverging. The contribution from such a pair of reversal-diverging paths will cancel out the contributions from $(n-1)$ pairs of transversal-divering paths but, again, only if the axes of the torus are of equal length.

We turn next to another curious property of the hodotic solutions, which becomes useful in decomposing solutions of the wave equation into components other than the $2n$ travelling waves discussed above. Let *promotion* of an m-dimensional solution of (2) to an n-dimensional space, with $n>m$, denote the following procedure.

Procedure. Take the (m-dimensional) hodotic solution at any two successive time values and imbed it in a space of dimension n. That is, use the solution as the boundary conditions on a hyperplane of dimension m in an n-dimensional space, with the

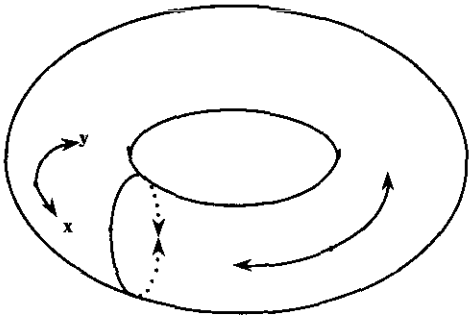

Figure 7. If the two axes of the torus are of unequal length, say $L_y > L_x$, then, at $t = L_x/2+1 \equiv T$, a pair of paths forming a loop along the x-axis will not be cancelled by a corresponding loop along the y-direction, since such loops could only be made of paths of length greater than T.

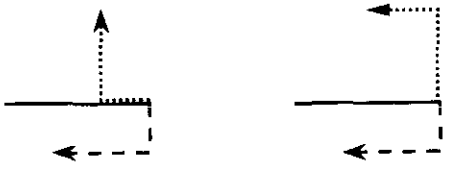

Figure 8. The pair of paths on the left diverges with one path making a reversal; it is said to be reversal diverging. The pair on the right diverges with neither of the paths making a reversal; it is said to be transversal diverging. (The full line indicates the portions of the two paths that are identical.)

solution being zero everywhere outside the hyperplane. Afterwards, iterate according to (2) (the speed of light becoming $\sqrt{1/n}$).

The hodotic solutions have the interesting property that they may be itereated an arbitrary number of times, promoted too a higher dimension, iterated again for some other arbitrary number of times, promoted to a still higher-dimensional space, and so on, all the while producing a solution whose discrete integrals over the corresponding space of the point solution, as well as its square, are constant. The proof is similar to the one outlined previously; classes of pairs of non-identical paths can again be shown to cancel although, for the purposes of the proof, the number of classes to which a pair of paths may belong varies according to the time at which the two members of the pair diverged (more precisely, according to whether they diverged before or after the given imbedding in the higher dimensional space). Note that, in general, this result does not hold for a torus.

Finally, note that the n-dimensional hodotic solutions, are simply the promotions of one-dimensional hodotic solutions to the appropriate higher dimension.

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